# On the transition from two-dimensional to three-dimensional MHD turbulence

**By   A. Thess † and Oleg Zikanov ‡**

We report a theoretical investigation of the robustness of two-dimensional inviscid MHD flows at low magnetic Reynolds numbers with respect to three-dimensional perturbations. We analyze three model problems, namely the flow in the interior of a triaxial ellipsoid, an unbounded vortex with elliptical streamlines, and a vortex sheet parallel to the magnetic field. We demonstrate that motion perpendicular to the magnetic field with elliptical streamlines becomes unstable with respect to the elliptical instability once the velocity has reached a critical magnitude whose value tends to zero as the eccentricity of the streamlines becomes large. Furthermore, vortex sheets parallel to the magnetic field, which are unstable for any velocity and any magnetic field, are found to emit eddies with vorticity perpendicular to the magnetic field and with an aspect ratio proportional to $N^{1/2}$. The results suggest that purely two-dimensional motion without Joule energy dissipation is a singular type of flow which does not represent the asymptotic behavior of three-dimensional MHD turbulence in the limit of infinitely strong magnetic fields.

## 1. Introduction

Turbulent flows of liquid metals influenced by magnetic fields occur under a wide range of circumstances, ranging from metallurgy (Davidson 1999) and fundamental turbulence research (Tsinober 2001; Moresco & Alboussiere 2004) to the movement of the Earth's inner core (Moffatt 1978). It is widely believed that when the magnetic Reynolds number $Rm = \mu_0 \sigma U L$ is small, and the magnetic field is sufficiently strong, homogeneous MHD-turbulence becomes purely two-dimensional and the electromagnetic dissipation of kinetic energy vanishes. The purpose of the present work is to demonstrate that this belief often contradicts to reality, and that homogeneous MHD-turbulence may differ strongly from purely two-dimensional turbulence even when the magnetic field tends to infinity.

When an incompressible fluid with density $\rho$, kinematic viscosity $\nu$, electrical conductivity $\sigma$, and permeability $\mu_0$ moves with velocity $U$ and characteristic length scale $L$ in a uniform magnetic field $B$, and when Rm $\ll$ 1, the state of the flow is characterized by the Reynolds number and the electromagnetic interaction parameter defined as

$$Re = \frac{UL}{\nu}, \quad N = \frac{\sigma B^2 L}{\rho U}. \tag{1.1}$$

We are interested in fully developed turbulence ($Re \to \infty$) far away from walls under a strong magnetic field $N >> 1$. It has been established by Moffatt (1967) (see also Davidson 1997) and confirmed experimentally by Alemany *et al* (1979) and numerically by Schumann (1976), Zikanov & Thess (1998), Knaepen & Moin (2004) that the turbulent flow tends to become two-dimensional so as to avoid electromagnetic (Joule) dissipation. Sommeria & Moreau (1982), who identified the importance of walls perpendicular to the

† Ilmenau University of Technology
‡ University of Michigan - Dearborn
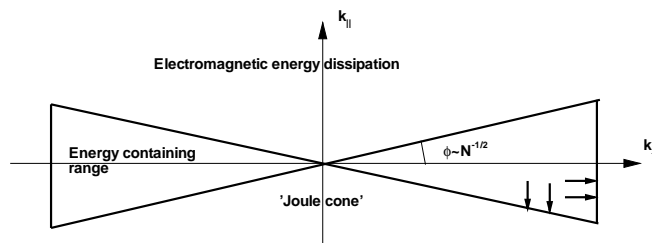
FIGURE 1. Wavenumber distribution of kinetic energy at $N \gg 1$.

magnetic field, have aptly summarized our conceptual understanding of MHD turbulence at $N \gg 1$ by characterizing the distribution of kinetic energy among the wavenumbers $k_\perp$ and $k_\parallel$ perpendicular and parallel to the magnetic field, respectively. This distribution of kinetic energy in the state of so called "quasi-two-dimensional turbulence" is sketched in figure 1. For $N \gg 1$ the kinetic energy of the flow is confined to a narrow region in the wavenumber space, outside the "Joule cone" in which electromagnetic energy dissipation takes place and below those wavenumbers $k_\perp$ for which viscous dissipation dominates. The angle $\varphi \sim N^{-1/2}$ of the energy containing region shrinks to zero as $N \to \infty$. This picture has tempted researchers to assume that for sufficiently strong magnetic field quasi-two-dimensional MHD turbulence becomes purely two-dimensional and the Joule energy dissipation vanishes.

However, there have been strong experimental and numerical indications which suggest that this hypothesis is unlikely to be correct. In their experiments on freely decaying homogeneous MHD turbulence in mercury Alemany *et al.* (1979) and Caperan & Alemany (1985) found that velocity fluctuations parallel to the applied magnetic field and strong Joule dissipation persisted even at high interaction parameters. Disagreements between drag measurements of MHD flows in channels with very large aspect ratios (low influence of Hartmann walls) (Tsinober 1990) and purely two-dimensional numerical simulations of turbulent channel flow by Jimenez (1990) further support the view that quasi-two-dimensional turbulence may never become purely-two-dimensional. Direct numerical simulations of forced MHD turbulence by Zikanov & Thess (1998) indicated that two-dimensional vortices, once they have formed, may undergo violent three-dimensional instabilities resulting in an intermittent behavior. Finally, numerical simulations of MHD turbulence subject to a two-dimensional forcing, which have been carried out by Nakauchi *et al.* (1992) using an EDQNM model, have provided evidence that even for very high values of the interaction parameter and strong non-isotropy of the flow, the Joule energy dissipation of the flow tends to a nonzero finite value.

The foregoing observations lead us to the conclusion that purely two-dimensional turbulence with zero Joule dissipation is a singular state which may never be reached in MHD, similarly as ordinary hydrodynamic turbulence in the limit of vanishing viscosity will never come close to a state of zero viscous dissipation. We may formulate the following hypothesis: When an electrically conducting fluid is in the state of fully developed turbulence ($\nu \to 0$) the Joule dissipation of kinetic energy will remain finite as $B \to \infty$.

In this paper we investigate the singular character of purely two-dimensional turbulence using stability analysis of model problems. We shall demonstrate that for arbitrarily strong magnetic fields a two-dimensional inviscid flow has always access to a sufficient number of unstable degrees of freedom to escape from purely two-dimensional behavior.

## 2. A didactical model for the instability of purely two-dimensional MHD flow

### 2.1. *An exact solution for inviscid MHD flow in a triaxial ellipsoid*

Consider an inviscid electrically conducting incompressible fluid which is confined to the interior of the triaxial ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{2.1}$$

and subjected to the influence of a homogeneous magnetic field $\mathbf{B} = B\mathbf{e}_z$. If the magnetic Reynolds number is small, the dynamics of the flow is governed by the equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{1}{\rho}\mathbf{J} \times \mathbf{B}, \quad \nabla \cdot \mathbf{v} = 0 \tag{2.2}$$

$$\mathbf{J} = \sigma(-\nabla \phi + \mathbf{v} \times \mathbf{B}), \quad \nabla \cdot \mathbf{J} = 0 \tag{2.3}$$

where $\mathbf{J}$ is the electric current density and $\phi$ the electric potential (Roberts 1967; Moreau 1990; Davidson 2001). The equations are supplemented by the boundary conditions at impermeable electrically insulating walls of the ellipsoid

$$\mathbf{v} \cdot \mathbf{n} = \mathbf{J} \cdot \mathbf{n} = 0. \tag{2.4}$$

If there is no magnetic field, there exists a family of three-dimensional time-dependent exact solutions of the Euler equations

$$\mathbf{v}(x,y,z,t) = U(t)\left[y\frac{c}{b}\mathbf{e}_z - z\frac{b}{c}\mathbf{e}_y\right] + V(t)\left[z\frac{a}{c}\mathbf{e}_x - x\frac{c}{a}\mathbf{e}_z\right] + W(t)\left[x\frac{b}{a}\mathbf{e}_y - y\frac{a}{b}\mathbf{e}_x\right], \tag{2.5}$$

where $(U, V, W)$ satisfy the nonlinear ordinary differential equations, which are mathematically identical to the equations describing the free rotation of a solid body (Kerswell 2002). The velocity field described by (2.5) automatically satisfies the incompressibility constraint, the free-slip condition, and is a superposition of three basic two-dimensional flows. Each of them has elliptical streamlines and a spatially uniform vorticity directed along the axis of the ellipsoid which is perpendicular to the plane of motion.

We extend the existing theory to the case when a magnetic field is present by expressing the electric current density in the form

$$\mathbf{J}(x,y,z,t) = I(t)\left[y\frac{c}{b}\mathbf{e}_z - z\frac{b}{c}\mathbf{e}_y\right] + J(t)\left[z\frac{a}{c}\mathbf{e}_x - x\frac{c}{a}\mathbf{e}_z\right] + K(t)\left[x\frac{b}{a}\mathbf{e}_y - y\frac{a}{b}\mathbf{e}_x\right] \tag{2.6}$$

which satisfies the condition $\nabla \cdot \mathbf{J} = 0$ as well as the boundary condition. The coefficients $I, J, K$ describe the strength of eddy currents induced by the movement of the fluid in the magnetic field. Taking the curl of Ohm's law (2.3) we obtain, after some algebra, $I = \sigma BV ab/(b^2 + c^2)$, $J = -\sigma BU ab/(a^2 + c^2)$, $K = 0$. The vorticity $\boldsymbol{\Omega} = \nabla \times \mathbf{v}$ corresponding to the velocity field (2.5)

$$\boldsymbol{\Omega} = U(t)\left(\frac{b}{c} + \frac{c}{b}\right)\mathbf{e}_x + V(t)\left(\frac{a}{c} + \frac{c}{a}\right)\mathbf{e}_y + W(t)\left(\frac{a}{b} + \frac{b}{a}\right)\mathbf{e}_z \tag{2.7}$$

satisfies exactly the vorticity equation

$$\frac{\partial \boldsymbol{\Omega}}{\partial t} = (\boldsymbol{\Omega} \cdot \nabla)\mathbf{v} + \frac{1}{\rho}\nabla \times (\mathbf{J} \times \mathbf{B}), \tag{2.8}$$

provided that the coefficients $U, V, W$ obey the equations

$$(b^2 + c^2)\dot{U} = (b^2 - c^2)VW - \frac{\sigma B^2}{\rho}\left(\frac{a^2 b^2}{a^2 + c^2}\right)U \tag{2.9}$$

$$(c^2 + a^2)\dot{V} = (c^2 - a^2)UW - \frac{\sigma B^2}{\rho}\left(\frac{a^2 b^2}{b^2 + c^2}\right)V \tag{2.10}$$

$$(a^2 + b^2)\dot{W} = (a^2 - b^2)UV \tag{2.11}$$

It should be emphasized that the derivation of this system from the governing equations (2.2) and (2.3) did not involve any approximation.

The system (2.9)-(2.11) can be made nondimensional by introducing dimensionless variables according to $(U, V, W) = \alpha(U_*, V_*, W_*)$ and $t = t_*/\alpha$ where
$\alpha = (\sigma B^2/\rho)a^2 b^2/[(a^2 + c^2)(b^2 + c^2)]$ is the inverse of the Joule decay time. Using the abbreviations $A = a/c$ and $B = b/c$ and dropping the asterisk the equations become

$$\dot{U} = \frac{B^2 - 1}{B^2 + 1}\,VW - U \tag{2.12}$$

$$\dot{V} = \frac{1 - A^2}{1 + A^2}\,UW - V \tag{2.13}$$

$$\dot{W} = \frac{A^2 - B^2}{A^2 + B^2}\,UV \tag{2.14}$$

This nonlinear system has a number of remarkable properties. The nonlinear terms conserve the total kinetic energy and the total angular momentum

$$E = \frac{1}{2}\left[(1 + B^2)U^2 + (1 + A^2)V^2 + (A^2 + B^2)W^2\right] \tag{2.15}$$

$$L = (1 + B^2)^2 U^2 + (1 + A^2)^2 V^2 + (A^2 + B^2)^2 W^2 \tag{2.16}$$

The magnetic field gives rise to an anisotropic damping, embodied in the linear dissipative terms on the right-hand-side of equations (2.12) and (2.13). The total kinetic energy obeys

$$\frac{dE}{dt} = -\epsilon \tag{2.17}$$

where $\epsilon = (1 + B^2)U^2 + (1 + A^2)V^2$ is the rate of Joule dissipation. As a result, the kinetic energy of the flow will always decay unless it is in a purely two-dimensional motion perpendicular to the magnetic field ($U = V = 0, W \neq 0$). Observe that the electromagnetic interaction parameter N (cf. equation (1.1)) does not appear in the system because we have used the Joule timescale $\alpha^{-1}$ to nondimensionalize the velocity. Instead, the values of U, V, and W represent the order of magnitude of N.

### 2.2. *Linear stability of steady states*

Equations (2.12)-(2.14) admit a steady solution $(U, V, W) = (0, 0, W_0)$ describing two-dimensional motion perpendicular to the magnetic field. We analyze the linear stability of this solution by solving the equations for infinitesimal perturbations $(\xi, \eta, \zeta)$

$$\dot{\xi} = -\xi + \left(\frac{B^2 - 1}{B^2 + 1}W_0\right)\eta \tag{2.18}$$

$$\dot{\eta} = \left(\frac{1 - A^2}{1 + A^2}W_0\right)\xi - \eta \tag{2.19}$$

$$\dot{\zeta} = 0. \tag{2.20}$$

The solvability condition yields the growth rates $\lambda$ in the form

$$\lambda_{1/2} = \pm W_0 \sqrt{\frac{(A^2 - 1)(1 - B^2)}{(A^2 + 1)(1 + B^2)} - 1}, \quad \lambda_3 = 0 \tag{2.21}$$

Instability occurs when $A > 1, B < 1$ or $A < 1, B > 1$, i.e. when the magnetic field is parallel to the middle axis of the ellipsoid and the $W$-component exceeds the critical value

$$W_c = \sqrt{\frac{(A^2 + 1)(1 + B^2)}{(A^2 - 1)(1 - B^2)}} \text{ or, in dimensional form, } W_c^{dim} = \frac{\sigma B^2}{\rho} \sqrt{\frac{(a^2 + c^2)(c^2 + b^2)}{(a^2 - c^2)(c^2 - b^2)}} \tag{2.22}$$

When $a \to \infty$ and $b \to 0$ we have $W_c \to 1$ and $W_c^{dim} \to \sigma B^2/\rho$ which implies that a very elongated sheet-like structure becomes unstable when the interaction parameter defined as $N = \sigma B^2/\rho W$ is smaller than $N_c = 1$.

The instability that we have identified in this particular MHD problem is the elliptical instability, widely known in ordinary hydrodynamics as a fundamental mechanism for the transition to turbulence (see, e.g., Kerswell 2002). It is generated by parametric resonance between Kelvin waves, coming from the homogeneously rotating part of the velocity field, and the strain field, being a result of the elliptical shape of the streamlines. We have just shown that a magnetic field can damp but not completely suppress this instability.

### 2.3. *Nonlinear dynamics and the singular character of purely two-dimensional motion*

We now analyze the nonlinear evolution of the system focusing the attention on the case where the initial state is an almost two-dimensional flow characterized by $U(0) \ll W(0)$ and $V(0) \ll W(0)$. Solution of (2.12)-(2.14) for $A = 2$ and $B = 0.5$ shows that as long as $W(0) < W_c = 5/3$, the system always relaxes to a purely two-dimensional state $(U = 0, V = 0, W < W_c)$. Figure 2 shows that the evolution from an initial condition with $W(0) > W_c$ proceeds in a quite different way. Although the system finally settles at a purely two-dimensional state, the evolution towards this state is characterized by long periods of nearly two-dimensional motion, occasionally interrupted by violent three-dimensional transients. During these transients kinetic energy is fed from the nondissipative two-dimensional mode $W$ to the three-dimensional modes $U$ and $V$ as can be seen in figure 2c. The transients lead to significant Joule dissipation of kinetic energy as illustrated in figures 2b and 2d. Decay of initially almost two-dimensional MHD flow is accompanied by strong divergence from two-dimensionality due to the action of the elliptical instability.

The singular character of purely two-dimensional evolution is further illustrated by extending our model so as to include a phenomenological forcing term, which corresponds to application of an external torque in the $z$-direction. Equation (2.14) is replaced by

$$\dot{W} = \frac{A^2 - B^2}{A^2 + B^2} UV + 1 \tag{2.23}$$

The new model admits an exact two-dimensional solution

$$U = 0, \quad V = 0, \quad W = t \tag{2.24}$$

whose kinetic energy grows monotonically with time and whose Joule dissipation is zero. However, the numerical solution computed from a weakly three-dimensional initial condition shows a completely different behavior. The flow corresponding to the numerical
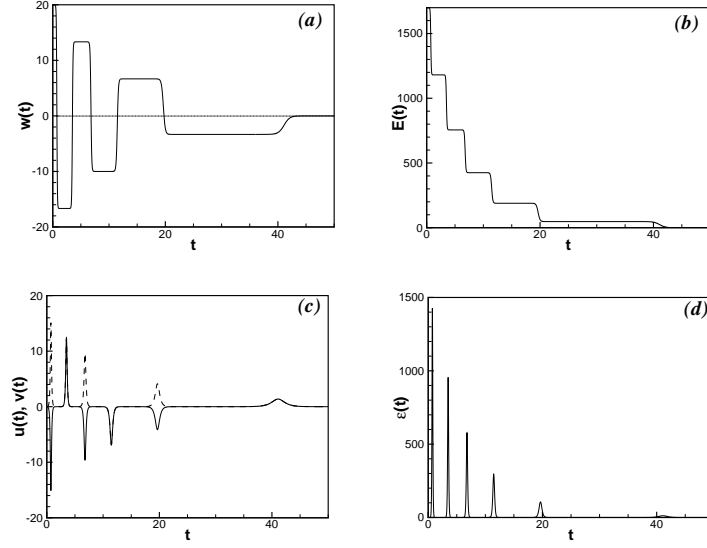
FIGURE 2. Decaying solution of (2.12-2.14). *(a)*, Two-dimensional component $W(t)$. *(b)*, Total kinetic energy (2.15). *(c)*, Three-dimensional components $U(t)$ (solid line) and $V(t)$ (dashed line). *(d)*. Rate of Joule dissipation $\epsilon$.
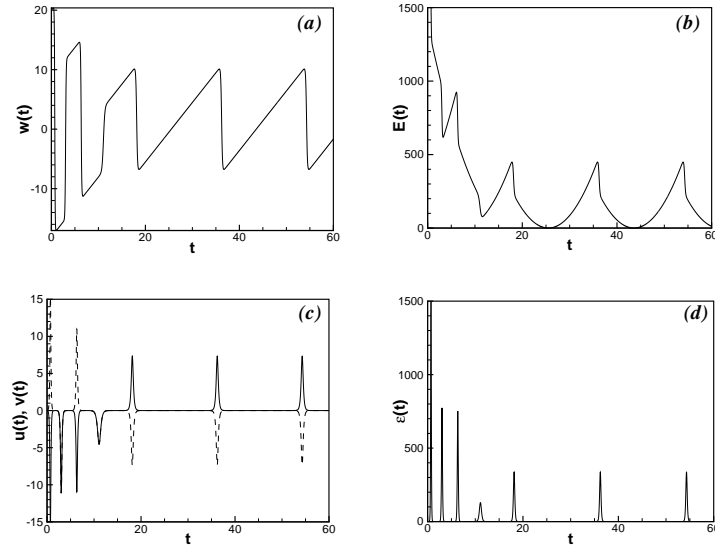


FIGURE 3. As in Figure 2 but for forced system (2.12, 2.13, 2.23).

solution shown in figure 3 is periodic in time and involves long lasting periods of two-dimensional motion interrupted by short three-dimensional excursions. Most importantly, the flow is "statistically" steady in that the time-averaged kinetic energy does not change and the mean Joule dissipation is finite. The flow is therefore neither mathematically nor physically close to the two-dimensional behavior suggested by the exact solution (2.24).

Although our simple model is only a caricature of the behaviour of real MHD turbulence, it allows us to draw a number of conclusions which serve as a guideline for the investigation of the full problem. In particular, we have learned

- that a purely two-dimensional MHD flow is prone to the elliptical instability,
- that the nonlinear three-dimensional evolution of the elliptical instability effectively extracts kinetic energy from the system on account of Joule dissipation,
- that an initially nearly two-dimensional flow does not necessarily stay close to purely two-dimensional evolution.

## 3. Instability in two-dimensional turbulence in a strong magnetic field

For further analysis it is convenient to rewrite (2.2) and (2.3) as (Roberts 1967)

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p - \frac{\sigma B^2}{\rho}\Delta^{-1}\frac{\partial^2 \mathbf{v}}{\partial z^2}, \quad \nabla \cdot \mathbf{v} = 0. \tag{3.1}$$

We are interested in the stability of two-dimensional flows for which the Joule dissipation vanishes. The most general flow of this type is of the form

$$\mathbf{v}(x, y, t) = \mathbf{U}(x, y, t) + W(x, y, t)\mathbf{e}_z \tag{3.2}$$

where $\mathbf{U}$ has the properties $\mathbf{U} \cdot \mathbf{e}_z = 0$, $\nabla \cdot \mathbf{U} = 0$ and $\mathbf{U}$ and $W$ are solutions of

$$\frac{\partial \mathbf{U}}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{U} = -\frac{1}{\rho}\nabla P, \quad \frac{\partial W}{\partial t} + (\mathbf{U} \cdot \nabla)W = 0 \tag{3.3}$$

The two terms in (3.2) describe the superposition of a two-dimensional motion perpendicular to the magnetic field with a motion parallel to the magnetic field.

Our hypothesis formulated in section 1 implies that the solution (3.2) of equation (3.1) is always unstable in the sense that for any initial condition containing a weak three-dimensional perturbation the solution will diverge from the two-dimensional one and evolve towards a solution with nonzero mean Joule dissipation no matter how strong the magnetic field is. A rigorous proof of this assertion is a formidable task, as it would require consideration of the initial value problem (3.1) for all admissible fields $\mathbf{U}$ and $\mathbf{V}$. We shall simplify the task considerably by considering two families of structures which we believe are representative of the solutions to (3.3) namely, columnar vortices with elliptical streamlines and vortex sheets whose velocity is parallel to the direction of the magnetic field.

### 3.1. *Instability of motion perpendicular to the magnetic field*

We consider an unbounded strained vortex described by

$$\mathbf{U}(x, y) = -\Omega E y \mathbf{e}_x + \Omega E^{-1} x \mathbf{e}_y \tag{3.4}$$

When the eccentricity $E = 1$, the flow is a solid body rotation around the axis of the magnetic field with vorticity $2\Omega$. When $E > 1$, the flow has a uniform strain and its streamlines are ellipses with semi-axes $a = \sqrt{E}$ and $b = 1/\sqrt{E}$. We extend the analysis of Bayly (1986) and Waleffe (1990) to the magnetic case by perturbing the basic flow (3.4) according to $\mathbf{v} = \mathbf{U} + \mathbf{u}$. Linearizing (3.1) with respect to the perturbation we obtain

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{U} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{U} = -\frac{1}{\rho}\nabla p - \alpha\Delta^{-1}\frac{\partial^2 \mathbf{u}}{\partial z^2}, \quad \nabla \cdot \mathbf{u} = 0. \tag{3.5}$$

where $\alpha = \sigma B^2/\rho$. This system admits solutions of the form

$$\{\mathbf{u},\, p\} = \{\hat{\mathbf{u}},\, \hat{p}\}(t) \cdot \exp\left[i\mathbf{k}(t) \cdot \mathbf{x} - \alpha \int_{t_0}^{t} \frac{k_z^2(t')}{k^2(t')} dt'\right] \tag{3.6}$$

Inserting (3.6) into (3.5) we obtain equations identical to those for the nonmagnetic problem solved by Bayly (1986) and Waleffe (1990).

$$\dot{\hat{u}}_j + i\dot{k}_l x_l \hat{u}_j + ik_m A_{ml} x_l \hat{u}_j + A_{jl}\hat{u}_l = -ik_j\hat{p}, \quad k_j\hat{u}_j = 0, \tag{3.7}$$

where $A_{12} = -\Omega E$, $A_{21} = \Omega/E$ and $A_{ij} = 0$ otherwise. The solution includes the wavenumber vector $\mathbf{k}$ rotating around the axis of the vortex as

$$\mathbf{k}(t) = k_0[\sin\theta\cos\phi(t), E\sin\theta\sin\phi(t), \cos\theta] \tag{3.8}$$

where $\phi(t) = \Omega(t - t_0)$, and the Floquet solution $\hat{\mathbf{u}}(t) = e^{\lambda t}\mathbf{w}\,[\phi(t)]$, where $\mathbf{w}$ is a $2\pi$-periodic function and the growth rate is

$$\lambda(\Omega, E, \theta) = \frac{\Omega}{2\pi}\ln[\mu(E, \theta)]. \tag{3.9}$$

Here, $\mu$ are eigenvalues of an auxiliary eigenvalue problem. The amplitude of perturbations is

$$|\mathbf{u}(t)| = |\mathbf{w}(t)| \cdot \exp[F(t)] \tag{3.10}$$

with

$$F(t) = \lambda(E, \theta, \Omega)t - \alpha\int_{t_0}^{t} \frac{\cos^2\theta\, dt'}{\sin^2\theta\cos^2\Omega(t' - t_0') + E\sin^2\theta\cos^2\Omega(t' - t_0') + \cos^2\theta}. \tag{3.11}$$

The second term in the right-hand-side is the only correction to the non-magnetic solution. Reverting to dimensionless time $\phi = \Omega(t - t_0)$ and using a magnetic interaction parameter defined by

$$N = \alpha/\Omega = \sigma B^2/\rho\Omega \tag{3.12}$$

we can express $F$ as

$$F(\phi) = \frac{\phi}{2\pi}\ln[\mu(E, \theta)] - N\int_0^{\phi} \frac{d\phi'}{1 + \tan^2\theta[\cos^2\phi' + E\sin^2\phi']}. \tag{3.13}$$

The system is unstable if F increases over one period, i.e. $F(2\pi) - F(0) > 0$, it is stable if $F(2\pi) - F(0) < 0$ and the condition for neutral stability is therefore $F(2\pi) - F(0) = 0$. From the last condition we obtain the desired expression for the neutral surface as

$$N(E, \theta) = \ln[\mu(E, \theta)] \cdot \left\{\int_0^{2\pi} \frac{d\phi}{1 + \tan^2\theta[\cos^2\phi + E\sin^2\phi]}\right\}^{-1} \tag{3.14}$$

Figure 4a shows the numerically computed values of (3.14). For $E = 1$ we have $N = 0$ indicating that a vortex with circular stream lines is always stable. The modes take the form of Kelvin waves which rotate about the magnetic field with a frequency $2\Omega\cos\theta$ (see e.g. Greenspan 1968). At $E > 1$, the flow becomes unstable with respect to perturbations located within a band $\theta_-(E) \le \theta \le \theta_+(E)$. Notice that the location of this unstable band is the same as in the nonmagnetic problem since it is determined by the zeros of $\ln[\mu(E, \theta)]$. In particular, this band originates at $\theta = \pi/3$ (Bayly 1986). For a given value of $E > 1$ the maximum of $N(E, \theta)$ over all $\theta$ in the unstable band defines the critical magnetic interaction parameter $N_c(E)$ and a critical angle $\theta_c(E)$. These quantities are
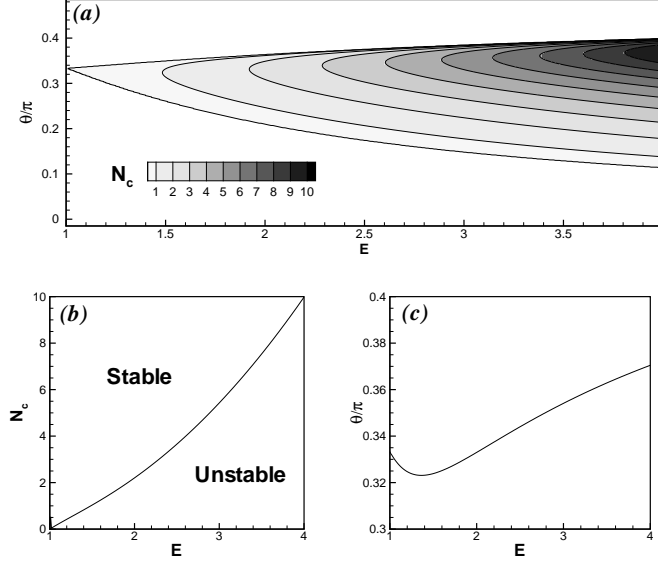
FIGURE 4. Stability of an unbounded strained vortex. *(a)*, Neutral magnetic interaction parameter (3.14) as a function of eccentricity $E$ and angle $\theta$ between the wavenumber vector **k** of the perturbations and the axis of the vortex. *(b)* and *(c)*, critical interaction parameter $N_c$ and $\theta_c$ as functions of $E$.

plotted in figure 4b and c. Figure 4b shows that $N_c$ increases rapidly with increasing eccentricity while the critical angle $\theta_c$ stays in the vicinity of $\pi/3$. The monotonic increase of the critical interaction parameter implies that strongly elliptical (almost sheet-like) structures are particularly prone to instability. Having in mind that the nonlinear evolution of initially circular vortices governed by the two-dimensional Euler equation (3.2) always leads to the generation of strongly elongated vortex sheets, we can conclude that the two-dimensional evolution proceeds in a way as to make the system more susceptible to the elliptical instability. The fact that $\theta_c \sim \pi/3$ may seem counterintuitive, as one usually expects in MHD that unstable structures have a tendency to align with the magnetic field ($\theta \to \pi/2$). However, it should be noticed that $\theta_c$ is monotonically increasing and probably asymptotes towards $\pi/2$.

Our stability result can be also viewed from another perspective. In a given magnetic field any forced two-dimensional flow will eventually become unstable, once the vorticity amplitude has reached a sufficiently high level $\Omega$ such that N falls below its critical value and the system becomes three-dimensionally unstable.

### 3.2. *Instability of motion parallel to the magnetic field*

In purely two-dimensional flow the velocity component parallel to the magnetic field behaves like a passive scalar, transported by the velocity $\mathbf{U}(x,y,t)$, cf. equation (3.2). Such evolution is known to produce steep gradients of $W(x,y,t)$ [see e.g. Kraichnan & Montgomery (1980), Lesieur (1990)]. Since the archetype of such structures is a single vortex sheet, which is known to undergo a Kelvin-Helmholtz instability, we will discuss the stability of the flow

$$W(x) = U_0 \frac{x}{|x|} \tag{3.15}$$

under the influence of the magnetic field. This problem, along with more general unidirectional velocity fields, was originally considered by Drazin (1960). However, the spatial structure of the unstable modes were not investigated there and is therefore the central focus of the discussion given below. Infinitesimal perturbations $\mathbf{u}$ superimposed upon the basic flow (3.15) are governed by the linearized equation

$$\frac{\partial \mathbf{u}}{\partial t} + W\frac{\partial \mathbf{u}}{\partial z} + u_x \frac{dW}{dx}\mathbf{e}_z = -\frac{1}{\rho}\nabla p - \alpha\Delta^{-1}\frac{\partial^2 \mathbf{u}}{\partial z^2}, \quad \nabla \cdot \mathbf{u} = 0. \qquad (3.16)$$

Squire's theorem, which holds for flow in a parallel magnetic field (Michael 1953; Stuart 1954), permits one to reduce the analysis to two-dimensional perturbations of the form $\mathbf{u} = (u, 0, w)$. They can be decomposed into normal modes according to

$$u(x, z, t) = \hat{u}(x) \cdot \exp[ik(z - ct)], \dots \qquad (3.17)$$

After introducing a stream function $\hat{\psi}$ via $\hat{u} = ik\hat{\psi}$ and $\hat{w} = -d\hat{\psi}/dx$, nondimensional variables via $U = U_0 U_*$, $x = x_*/k$, $c = U_0 c_*$, $t = t' * /U_0 k$, an interaction parameter $N = \sigma B^2/\rho U_0 k$, and dropping the asterisk, it can be shown (Drazin 1960) that the complex wave velocity $c$ is determined by the third-order polynomial equation

$$-ic(1 + c^2) + \frac{N}{4}(1 + 3c^2) = 0. \qquad (3.18)$$

The normal mode for $x > 0$ is given by

$$\hat{\psi}(x) = \exp\left(-x\sqrt{1 - i\frac{N}{c-1}}\right) \qquad (3.19)$$

where it is assumed that the square root with the positive real part has been taken.

Drazin (1960) has demonstrated that all three roots of equation (3.18) are purely imaginary and that one of the solutions has always positive imaginary part, but he did not explicitly calculate $c(N)$ and $\psi(x)$. Figure 5a shows the imaginary parts of the three roots. The wavenumber $k$ does not explicitly appear in the solution because it represents the only characteristic length scale of the problem and has therefore been used to nondimensionalize the equations. However, the wavenumber enters the definition of N.

For each value of N there are two stable and one unstable solutions. When N=0 we recover the classical Kelvin-Helmholtz instability with $c = i$. For $N > 0$ the imaginary part of the solution originating from $c = i$ always remains positive which implies that the vortex sheet is unstable for arbitrarily strong magnetic fields. The modes with small N, which correspond to small perturbation wavelengths and small magnetic fields are growing fastest, while the modes with large N, corresponding to large scales and strong magnetic fields grow slowest. The asymptotic behavior of the wave velocity for the unstable mode is $c = i(1 - N/4)$ for $N \ll 1$ and $c = i/\sqrt{(3)}$ for $N \gg 1$. The spatial structure of the unstable modes is best revealed by the stream function $\psi(x, z, t) = Re\{\hat{\psi}(x) \cdot \exp[i(z - ct)]\}$ (nondimensional variables) which, for $t = 0$, is (see figures 5b and 5c)

$$\psi(x, z) = \exp\left[-\left(1 + \frac{N}{4}\right)x\right]\cos\left[-\frac{N}{4}x + z\right] \quad \text{for } N \ll 1 \quad (3.20)$$

$$\psi(x, z) = \exp\left[-3^{1/4}\left(\frac{N}{8}\right)^{1/2}x\right]\cos\left[-3^{3/4}\left(\frac{N}{8}\right)^{1/2}x + z\right] \quad \text{for } N \gg 1 \quad (3.21)$$
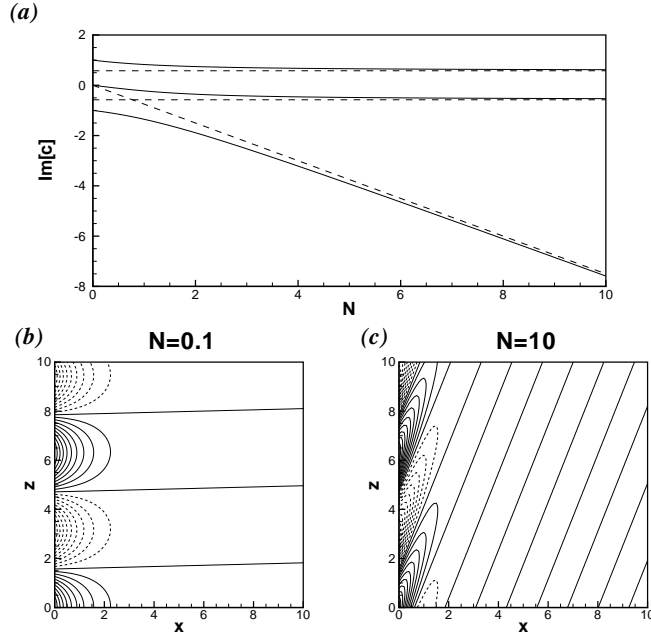
*(a)*



*(b)* N=0.1    *(c)* N=10



FIGURE 5. Instability of a vortex sheet. *(a)*, Solid lines - roots of the dispersion equation (3.18). Dashed lines - asymptotic limits $\pm 1/3^{1/2}$, $-3N/4$. *(b)*, Streamfunction of the unstable mode at $N \ll 1$ (3.20). *(c)*, Streamfunction of the unstable mode at $N \gg 1$ (3.21).

The unstable modes for large $N$ have an aspect ratio $z/x \sim N^{1/2}$ as is characteristic of MHD flows in strong magnetic fields.

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